

# Irreducibility and cuspidality

Dinakar Ramakrishnan\*

## Introduction

*Irreducible* representations are the building blocks of general, semisimple Galois representations  $\rho$ , and *cuspidal* representations are the building blocks of automorphic forms  $\pi$  of the general linear group. It is expected that when an object of the former type is associated to one of the latter type, usually in terms of an identity of  $L$ -functions, the irreducibility of the former should imply the cuspidality of the latter, and vice-versa. It is not a simple matter - *at all* - to prove this expectation, and nothing much is known in dimensions  $> 2$ . We will start from the beginning and explain the problem below, and indicate a result (in one direction) at the end of the Introduction, which summarizes what one can do at this point. The remainder of the paper will be devoted to showing how to deduce this result by a synthesis of known theorems and some new ideas. We will be concerned here only with the *so called easier* direction of showing the cuspidality of  $\pi$  given the irreducibility of  $\rho$ , and refer to [Ra5] for a more difficult result going the other way, which uses crystalline representations as well as a refinement of certain deep modularity results of Taylor, Skinner-Wiles, et al. Needless to say, *easier* does not mean easy, and the significance of the problem stems from the fact that it does arise (in this direction) naturally. For example,  $\pi$  could be a functorial, automorphic image  $r(\eta)$ , for  $\eta$  a cuspidal automorphic representation of a product of smaller general linear groups:  $H(\mathbb{A}) = \prod_j GL(m_j, \mathbb{A})$ , with an associated Galois representation  $\sigma$  such that  $\rho = r(\sigma)$  is irreducible. If the automorphy of  $\pi$  has been established by using a flexible converse theorem ([CoPS1]),

---

\*Partially supported by the NSF through the grant DMS-0402044

then the cuspidality of  $\pi$  is not automatic. In [RaS], we had to deal with this question for cohomological forms  $\pi$  on  $\mathrm{GL}(6)$ , with  $H = \mathrm{GL}(2) \times \mathrm{GL}(3)$  and  $r$  the Kronecker product, where  $\pi$  is automorphic by [KSh1]. Besides, the main result (Theorem A below) of this paper implies, as a consequence, the cuspidality of  $\pi = \mathrm{sym}^4(\eta)$  for  $\eta$  defined by any non-CM holomorphic newform  $\varphi$  of weight  $\geq 2$  relative to  $\Gamma_0(N) \subset \mathrm{SL}(2, \mathbb{Z})$ , without appealing to the criterion of [KSh2]; here the automorphy of  $\pi$  is known by [K] and the irreducibility of  $\rho$  by [Ri].

Write  $\overline{\mathbb{Q}}$  for the field of all algebraic numbers in  $\mathbb{C}$ , which is an infinite, mysterious Galois extension of  $\mathbb{Q}$ . One could say that the central problem in algebraic Number theory is to understand this extension. *Class field theory*, one of the towering achievements of the twentieth century, helps us understand the *abelian* part of this extension, though there are still some delicate, open problems even in that well traversed situation.

Let  $\mathcal{G}_{\mathbb{Q}}$  denote the absolute Galois group of  $\mathbb{Q}$ , meaning  $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ . It is a profinite group, being the projective limit of finite groups  $\mathrm{Gal}(K/\mathbb{Q})$ , as  $K$  runs over number fields which are normal over  $\mathbb{Q}$ . For fixed  $K$ , the Tchebotarev density theorem asserts that every conjugacy class  $C$  in  $\mathrm{Gal}(K/\mathbb{Q})$  is the *Frobenius class* for an infinite number of primes  $p$  which are unramified in  $K$ . This shows the importance of studying the *representations* of Galois groups, which are intimately tied up with conjugacy classes. Clearly, every  $\mathbb{C}$ -representation, i.e., a homomorphism into  $\mathrm{GL}(n, \mathbb{C})$  for some  $n$ , of  $\mathrm{Gal}(K/\mathbb{Q})$  pulls back, via the canonical surjection  $\mathcal{G}_{\mathbb{Q}} \rightarrow \mathrm{Gal}(K/\mathbb{Q})$ , to a representation of  $\mathcal{G}_{\mathbb{Q}}$ , which is continuous for the profinite topology. Conversely, one can show that every *continuous*  $\mathbb{C}$ -representation  $\rho$  of  $\mathcal{G}_{\mathbb{Q}}$  is such a pull-back, for a suitable finite Galois extension  $K/\mathbb{Q}$ . E. Artin associated an  $L$ -function, denoted  $L(s, \rho)$ , to any such  $\rho$ , such that the arrow  $\rho \rightarrow L(s, \rho)$  is additive and inductive. He conjectured that for any non-trivial, irreducible, continuous  $\mathbb{C}$ -representation  $\rho$  of  $\mathcal{G}_{\mathbb{Q}}$ ,  $L(s, \rho)$ , is entire, and this conjecture is open in general. Again, one understands well the *abelian* situation, i.e., when  $\rho$  is a 1-dimensional representation; the kernel for such a  $\rho$  defines an abelian extension of  $\mathbb{Q}$ . By class field theory, such a  $\rho$  is associated to a character  $\xi$  of finite order of the idele class group  $\mathbb{A}^*/\mathbb{Q}^*$ ; here, being *associated* means they have the same  $L$ -function, with  $L(s, \xi)$  being the one introduced by Hecke, *albeit* in a different language. As usual, we are denoting by  $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$  the topological *ring of adeles*, with  $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ , and by  $\mathbb{A}^*$  its multiplicative group of *ideles*, which can be given the structure of a locally compact abelian

topological group with discrete subgroup  $\mathbb{Q}^*$ .

Now fix a prime number  $\ell$ , and an algebraic closure  $\overline{\mathbb{Q}_\ell}$  of the field of  $\ell$ -adic numbers  $\mathbb{Q}_\ell$ , equipped with an embedding  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ . Consider the set  $\mathcal{R}_\ell(n, \mathbb{Q})$  of continuous, semisimple representations

$$\rho_\ell : \mathcal{G}_\mathbb{Q} \rightarrow \mathrm{GL}(n, \overline{\mathbb{Q}_\ell}),$$

up to equivalence. The image of  $\mathcal{G}_\mathbb{Q}$  in such a representation is usually not finite, and the simplest example of that is given by the  $\ell$ -adic cyclotomic character  $\chi_\ell$  given by the action of  $\mathcal{G}_\mathbb{Q}$  on all the  $\ell$ -power roots of unity in  $\overline{\mathbb{Q}}$ . Another example is given by the 2-dimensional  $\ell$ -adic representation on all the  $\ell$ -power *division points* of an elliptic curve  $E$  over  $\mathbb{Q}$ .

The correct extension to the non-abelian case of the *idele class character*, which appears in class field theory, is the notion of an irreducible *automorphic representation*  $\pi$  of  $\mathrm{GL}(n)$ . Such a  $\pi$  is in particular a representation of the locally compact group  $\mathrm{GL}(n, \mathbb{A}_F)$ , which is a restricted direct product of the local groups  $\mathrm{GL}(n, \mathbb{Q}_v)$ , where  $v$  runs over all the primes  $p$  and  $\infty$  (with  $\mathbb{Q}_\infty = \mathbb{R}$ ). There is a corresponding factorization of  $\pi$  as a tensor product  $\otimes_v \pi_v$ , with all but a finite number of  $\pi_p$  being *unramified*, i.e., admitting a vector fixed by the maximal compact subgroup  $K_v$ . At the archimedean place  $\infty$ ,  $\pi_\infty$  corresponds to an  $n$ -dimensional, semisimple representation  $\sigma(\pi_\infty)$  of the real Weil group  $W_\mathbb{R}$ , which is a non-trivial extension of  $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$  by  $\mathbb{C}^*$ . Globally, by Schur's lemma, the center  $Z(\mathbb{A}) \simeq \mathbb{A}^*$  acts by a quasi-character  $\omega$ , which must be trivial on  $\mathbb{Q}^*$  by the automorphy of  $\pi$ , and so defines an idele class character. Let us restrict to the central case when  $\pi$  is essentially unitary. Then there is a (unique) real number  $t$  such that the twisted representation  $\pi_u := \pi(t) = \pi \otimes |\cdot|^t$  is unitary (with unitary central character  $\omega_u$ ). We are, by abuse of notation, writing  $|\cdot|^t$  to denote the quasi-character  $|\cdot|^t \circ \det$  of  $\mathrm{GL}(n, \mathbb{A})$ , where  $|\cdot|$  signifies the adelic absolute value, which is trivial on  $\mathbb{Q}^*$  by the Artin product formula. Roughly speaking, to say that  $\pi$  is automorphic means  $\pi_u$  appears (in a weak sense) in  $L^2(Z(\mathbb{A})\mathrm{GL}(n, \mathbb{Q}) \backslash \mathrm{GL}(n, \mathbb{A}), \omega_u)$ , on which  $\mathrm{GL}(n, \mathbb{A}_F)$  acts by right translations. A function  $\varphi$  in this  $L^2$ -space whose averages over all the horocycles are zero is called a *cuspidal form*, and  $\pi$  is called *cuspidal* if  $\pi_u$  is generated by the right  $\mathrm{GL}(n, \mathbb{A}_F)$ -translates of such a  $\varphi$ . Among the automorphic representations of  $\mathrm{GL}(n, \mathbb{A})$  are certain distinguished ones called *isobaric automorphic representations*. Any isobaric  $\pi$  is of the form  $\pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r$ , where each  $\pi_j$  is a cuspidal representation of  $\mathrm{GL}(n_j, \mathbb{A})$ , such that  $(n_1, n_2, \dots, n_r)$  is a partition of  $n$ , where  $\boxplus$  denotes

the Langlands sum (coming from his theory of *Eisenstein series*); moreover, every *constituent*  $\pi_j$  is unique up to isomorphism. Let  $\mathcal{A}(n, \mathbb{Q})$  denote the set of isobaric automorphic representations of  $\mathrm{GL}(n, \mathbb{A})$  up to equivalence. Every isobaric  $\pi$  has an associated  $L$ -function  $L(s, \pi) = \prod_v L(s, \pi_v)$ , which admits a meromorphic continuation and a functional equation. Concretely, one associates at every prime  $p$  where  $\pi$  is unramified, a conjugacy class  $A(\pi)$  in  $\mathrm{GL}(n, \mathbb{C})$ , or equivalently, an unordered  $n$ -tuple  $(\alpha_{1,p}, \alpha_{2,p}, \dots, \alpha_{n,p})$  of complex numbers so that

$$L(s, \pi_p) = \prod_{j=1}^n (1 - \alpha_{j,p} p^{-s})^{-1}.$$

If  $\pi$  is cuspidal and non-trivial,  $L(s, \pi)$  is entire; so is the incomplete one  $L^S(s, \pi)$  for any finite set  $S$  of places of  $\mathbb{Q}$ .

Now suppose  $\rho_\ell$  is an  $n$ -dimensional, semisimple  $\ell$ -adic representation of  $\mathcal{G}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  *corresponds* to an automorphic representation  $\pi$  of  $\mathrm{GL}(n, \mathbb{A})$ . We will take this to mean that there is a finite set  $S$  of places including  $\ell, \infty$  and all the primes where  $\rho_\ell$  or  $\pi$  is ramified, such that we have,

$$(0.1) \quad L(s, \pi_p) = L_p(s, \rho_\ell), \quad \forall p \notin S,$$

where the Galois Euler factor on the right is given by the characteristic polynomial of  $Fr_p$ , the Frobenius at  $p$ , acting on  $\rho_\ell$ . When (0.1) holds (for a suitable  $S$ ), we will write

$$\rho_\ell \leftrightarrow \pi.$$

A natural question in such a situation is to ask if  $\pi$  is cuspidal when  $\rho_\ell$  is irreducible, and *vice-versa*. It is certainly what is predicted by the general philosophy. However, proving it is another matter altogether, and positive evidence is scarce beyond  $n = 2$ .

One can answer this question in the affirmative, for any  $n$ , if one restricts to those  $\rho_\ell$  which have *finite* image. In this case, it also defines a continuous,  $\mathbb{C}$ -representation  $\rho$ , the kind studied by E. Artin ([A]). Indeed, the hypothesis implies the identity of  $L$ -functions

$$(0.2) \quad L^S(s, \rho \otimes \rho^\vee) = L^S(s, \pi \times \pi^\vee),$$

where the superscript  $S$  signifies the removal of the Euler factors at places in  $S$ , and  $\rho^\vee$  (resp.  $\pi^\vee$ ) denotes the contragredient of  $\rho$  (resp.  $\pi$ ). The  $L$ -function on the right is the Rankin-Selberg  $L$ -function, whose mirific properties have been established in the independent and complementary works of

Jacquet, Piatetski-Shapiro and Shalika ([JPSS], and of Shahidi ([Sh1,2]); see also [MW]. A theorem of Jacquet and Shalika ([JS1]) asserts that the *order of pole* at  $s = 1$  of  $L^S(s, \pi \times \pi^\vee)$  is 1 iff  $\pi$  is cuspidal. On the other hand, for any finite-dimensional  $\mathbb{C}$ -representation  $\tau$  of  $\mathcal{G}_{\mathbb{Q}}$ , one has

$$(0.3) \quad -\text{ord}_{s=1} L^S(s, \tau) = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{G}_{\mathbb{Q}}}(\underline{1}, \tau),$$

where  $\underline{1}$  denotes the trivial representation of  $\mathcal{G}_{\mathbb{Q}}$ . Applying this with  $\tau = \rho \otimes \rho^\vee \simeq \text{End}(\rho)$ , we see that the order of pole of  $L^S(s, \rho \otimes \rho^\vee)$  at  $s = 1$  is 1 iff the only operators in  $\text{End}(\rho)$  which commute with the  $\mathcal{G}_{\mathbb{Q}}$ -action are scalars, which means by Schur that  $\rho$  is irreducible. Thus, *in the Artin case,  $\pi$  is cuspidal iff  $\rho_\ell$  is irreducible.*

For general  $\ell$ -adic representations  $\rho_\ell$  of  $\mathcal{G}_{\mathbb{Q}}$ , the order of pole at the right edge is not well understood. When  $\rho_\ell$  comes from *arithmetic geometry*, i.e., when it is a Tate twist of a piece of the cohomology of a smooth projective variety over  $\mathbb{Q}$  which is cut out by algebraic projectors, an important *conjecture of Tate* asserts an analogue of (0.3) for  $\tau = \rho_\ell \otimes \rho_\ell^\vee$ , but this is unknown except in a few families of examples, such as those coming from the theory of *modular curves*, *Hilbert modular surfaces* and *Picard modular surfaces*. So one has to find a different way to approach the problem, which works at least in low dimensions.

The main result of this paper is the following:

**Theorem A** *Let  $n \leq 5$  and let  $\ell$  be a prime. Suppose  $\rho_\ell \leftrightarrow \pi$ , for an isobaric, algebraic automorphic representation  $\pi$  of  $GL(n, \mathbb{A})$ , and a continuous,  $\ell$ -adic representation  $\rho_\ell$  of  $\mathcal{G}_{\mathbb{Q}}$ . Assume*

- (i)  $\rho_\ell$  is irreducible
- (ii)  $\pi$  is odd if  $n \geq 3$
- (iii)  $\pi$  is semi-regular if  $n = 4$ , and regular if  $n = 5$

*Then  $\pi$  is cuspidal.*

Some words of explanation are called for at this point. An isobaric automorphic representation  $\pi$  is said to be *algebraic* ([Cl1]) if the restriction of  $\sigma(\pi_\infty)$  to  $\mathbb{C}^*$  is of the form  $\bigoplus_{j=1}^n \chi_j$ , with each  $\chi_j$  algebraic, i.e., of the form  $z \rightarrow z^{p_j} \bar{z}^{q_j}$  with  $p_j, q_j \in \mathbb{Z}$ . (We do not assume that our automorphic representations are unitary, and the arrow  $\pi_\infty \rightarrow \sigma(\pi_\infty)$  will be normalized

arithmetically.) For  $n = 1$ , an algebraic  $\pi$  is an idele class character of type  $A_0$  in the sense of Weil. One says that  $\pi$  is *regular* iff  $\sigma(\pi_\infty)|_{\mathbb{C}^*}$  is a direct sum of characters  $\chi_j$ , each occurring with *multiplicity one*. And  $\pi$  is *semi-regular* ([BHR]) if each  $\chi_j$  occurs with *multiplicity at most two*. Suppose  $\xi$  is a 1-dimensional representation of  $W_{\mathbb{R}}$ . Then, since  $W_{\mathbb{R}}^{\text{ab}} \simeq \mathbb{R}^*$ ,  $\xi$  is defined by a character of  $\mathbb{R}^*$  of the form  $x \rightarrow |x|^w \cdot \text{sgn}(x)^{a(\xi)}$ , with  $a(\xi) \in \{0, 1\}$ ; here  $\text{sgn}$  denotes the sign character of  $\mathbb{R}^*$ . For every  $w$ , let  $\sigma_\infty[\xi] := \sigma(\pi_\infty(\frac{1-n}{2}))[\xi]$  denote the isotypic component of  $\xi$ , which has dimension at most 2 (resp. 1) if  $\pi$  is semi-regular (resp. regular), and is acted on by  $\mathbb{R}^*/\mathbb{R}_+^* \simeq \{\pm 1\}$ . We will call a semi-regular  $\pi$  *odd* if for every character  $\xi$  of  $W_{\mathbb{R}}$ , the eigenvalues of  $\mathbb{R}^*/\mathbb{R}_+^*$  on the  $\xi$ -isotypic component are distinct. Clearly, any regular  $\pi$  is odd under this definition. See section 1 for a definition of this concept for any algebraic  $\pi$ , not necessarily semi-regular.

I want to thank the organizers, Jae-Hyun-Yang in particular, and the staff, of the *International Symposium on Representation Theory and Automorphic Forms* in Seoul, Korea, first for inviting me to speak there (during February 14 – 17, 2005), and then for their hospitality while I was there. The talk I gave at the conference was on a different topic, however, and dealt with my ongoing work with Dipendra Prasad on *selfdual representations*. I would also like to thank F. Shahidi for helpful conversations and the referee for his comments on an earlier version, which led to an improvement of the presentation. It is perhaps apt to end this introduction at this point by acknowledging support from the National Science Foundation via the grant DMS – 0402044.

## 1 Preliminaries

### 1.1 Galois Representations

For any field  $k$  with algebraic closure, denote by  $\mathcal{G}_k$  the *absolute Galois group* of  $\bar{k}$  over  $k$ . It is a projective limit of the automorphism groups of finite Galois extensions  $E/k$ . We furnish  $\mathcal{G}_k$  as usual with the *profinite topology*, which makes it a *compact, totally disconnected topological group*. When  $k = \mathbb{F}_p$ , there is for every  $n$  a unique extension of degree  $n$ , which is Galois, and  $\mathcal{G}_{\mathbb{F}_p}$  is isomorphic to  $\widehat{\mathbb{Z}} \simeq \lim_n \mathbb{Z}/n$ , topologically generated by the *Frobenius automorphism*  $x \rightarrow x^p$ .

At each prime  $p$ , let  $\mathcal{G}_p$  denote the local Galois group  $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$  with inertia subgroup  $I_p$ , which fits into the following exact sequence:

$$(1.1.1) \quad 1 \rightarrow I_p \rightarrow \mathcal{G}_p \rightarrow \mathcal{G}_{\mathbb{F}_p} \rightarrow 1.$$

The fixed field of  $\overline{\mathbb{Q}}_p$  under  $I_p$  is the *maximal unramified extension*  $\mathbb{Q}_p^{\text{ur}}$  of  $\mathbb{Q}_p$ , which is generated by all the roots of unity of order prime to  $p$ . One gets a natural isomorphism of  $\text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p)$  with  $\mathcal{G}_{\mathbb{F}_p}$ . If  $K/\mathbb{Q}$  is unramified at  $p$ , then one can lift the Frobenius element to a conjugacy class  $\varphi_p$  in  $\text{Gal}(K/\mathbb{Q})$ .

All the Galois representations considered here will be continuous and finite-dimensional. Typically, we will fix a prime  $\ell$ , and algebraic closure  $\overline{\mathbb{Q}}_\ell$  of the field  $\mathbb{Q}_\ell$  of  $\ell$ -adic numbers, and consider a continuous homomorphism

$$(1.1.2) \quad \rho_\ell : \mathcal{G}_{\mathbb{Q}} \rightarrow \text{GL}(V_\ell),$$

where  $V_\ell$  is an  $n$ -dimensional vector space over  $\overline{\mathbb{Q}}_\ell$ . We will be interested only in those  $\rho_\ell$  which are unramified only at a finite set  $S$  of primes. Then  $\rho_\ell$  factors through a representation of the quotient group  $\mathcal{G}_S := G(\mathbb{Q}_S/\mathbb{Q})$ , where  $\mathbb{Q}_S$  is the maximal extension of  $\mathbb{Q}$  which is unramified outside  $S$ . One has the Frobenius classes  $\phi_p$  in  $\mathcal{G}_S$  for all  $p \notin S$ , and this allows one to define the  $L$ -factors (with  $s \in \mathbb{C}$ )

$$(1.1.3) \quad L_p(s, \rho_\ell) = \det(I - \varphi_p p^{-s} | V_\ell)^{-1}.$$

Clearly, it is the reciprocal of a polynomial in  $p^{-s}$  of degree  $n$ , with constant term 1, and it depends only on the equivalence class of  $\rho_\ell$ . One sets

$$(1.1.4) \quad L^S(s, \rho_\ell) = \prod_{p \notin S} L_p(s, \rho_\ell).$$

When  $\rho_\ell$  is the trivial representation, it is unramified everywhere, and  $L^S(s, \rho_\ell)$  is none other than the *Riemann zeta function*. To define the *bad factors* at  $p$  in  $S - \{\ell\}$ , one replaces  $V_\ell$  in this definition by the subspace  $V_\ell^{I_p}$  of *inertial invariants*, on which  $\varphi_p$  acts.

We are primarily interested in *semisimple representations* in this article, which are direct sums of *simple* (or *irreducible*) representations. Given any representation  $\rho_\ell$  of  $\mathcal{G}_{\mathbb{Q}}$ , there is an associated *semisimplification*, denoted  $\rho_\ell^{\text{ss}}$ , which is a direct sum of the simple Jordan-Holder components of  $\rho_\ell$ . A *theorem of Tchebotarev* asserts the density of the Frobenius classes in the

Galois group, and since the local  $p$ -factors of  $L(s, \rho_\ell)$  are defined in terms of the *inverse roots* of  $\varphi_p$ , one gets the following standard, but useful result.

**Proposition 1.1.5** *Let  $\rho_\ell, \rho'_\ell$  be continuous,  $n$ -dimensional  $\ell$ -adic representations of  $\mathcal{G}_\mathbb{Q}$ . Then*

$$L^S(s, \rho_\ell) = L^S(s, \rho'_\ell) \implies \rho_\ell^{\text{ss}} \simeq \rho'^{\text{ss}}_\ell.$$

The Galois representations  $\rho_\ell$  which have *finite image* are special, and one can view them as continuous  $\mathbb{C}$ -representations  $\rho$ . Artin studied these deeply, and showed, using the results of Brauer and Hecke, that the corresponding  $L$ -functions admit meromorphic continuation and a functional equation of the form

$$(1.1.6) \quad L^*(s, \rho) = \varepsilon(s, \rho) L^*(1 - s, \rho^\vee),$$

where  $\rho^\vee$  denotes the contragredient representation on the dual vector space, where

$$(1.1.7) \quad L^*(s, \rho) = L(s, \rho) L_\infty(s, \rho),$$

with the *archimedean factor*  $L_\infty(s, \rho)$  being a suitable product (shifted) gamma functions. Moreover,

$$(1.1.8) \quad \varepsilon(s, \rho) = W(\rho) N(\rho)^{s-1/2},$$

which is an entire function of  $s$ , with the (non-zero)  $W(\rho)$  being called the *root number* of  $\rho$ . The scalar  $N(\rho)$  is an integer, called the *Artin conductor* of  $\rho$ , and the finite set  $S$  which intervenes is the set of primes dividing  $N(\rho)$ . The functional equation shows that  $W(\rho)W(\rho^\vee) = 1$ , and so  $W(\rho) = \pm 1$  when  $\rho$  is *selfdual* (which means  $\rho \simeq \rho^\vee$ ). Here is a useful fact:

**Proposition 1.1.9** ([T]) *Let  $\tau$  be a continuous, finite-dimensional  $\mathbb{C}$ -representation of  $\mathcal{G}_\mathbb{Q}$ , unramified outside  $S$ . Then we have*

$$-\text{ord}_{s=1} L^S(s, \tau) = \text{Hom}_{\mathcal{G}_\mathbb{Q}}(\mathbb{1}, \tau).$$

**Corollary 1.1.10** *Let  $\rho$  be a continuous, finite-dimensional  $\mathbb{C}$ -representation of  $\mathcal{G}_\mathbb{Q}$ , unramified outside  $S$ . Then  $\rho$  is *irreducible* iff the incomplete  $L$ -function  $L^S(s, \rho \otimes \rho^\vee)$  has a *simple pole* at  $s = 1$ .*



Indeed, if we set

$$(1.1.11) \quad \tau := \rho \otimes \rho^\vee \simeq \text{End}(\rho),$$

then Proposition 1.1.9 says that the *order of pole* of  $L(s, \rho \otimes \rho^\vee)$  at  $s = 1$  is the *multiplicity of the trivial representation* in  $\text{End}(\rho)$  is 1, i.e., iff the *commutant*  $\text{End}_{\mathcal{G}_{\mathbb{Q}}}(\rho)$  is one-dimensional (over  $\mathbb{C}$ ), which in turn is equivalent, by Schur's lemma, to  $\rho$  being irreducible. Hence the Corollary.

For general  $\ell$ -adic representations  $\rho_\ell$ , there is no known analogue of Proposition 1.1.9, though it is predicted to hold (at the right edge of absolute convergence) by a *conjecture of Tate* when  $\rho_\ell$  comes from *arithmetic geometry* (see [Ra4], section 1, for example). Tate's conjecture is only known in certain special situations, such as for *CM abelian varieties*. For the  $L$ -functions in Tate's set-up, say of motivic weight  $2m$ , one does not even know that they make sense at the *Tate point*  $s = m + 1$ , let alone know its order of pole there. Things get even harder if  $\rho_\ell$  does not arise from a geometric situation. One cannot work in too general a setting, and at a minimum, one needs to require  $\rho_\ell$  to have some good properties, such as being unramified outside a finite set  $S$  of primes. Fontaine and Mazur conjecture ([FoM]) that  $\rho_\ell$  is *geometric* if it has this property (of being unramified outside a finite  $S$ ) and is in addition *potentially semistable*.

## 1.2 Automorphic Representations

Let  $F$  be a number field with adele ring  $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$ , equipped with the adelic absolute value  $|\cdot| = |\cdot|_{\mathbb{A}}$ . For every algebraic group  $G$  over  $F$ , let  $G(\mathbb{A}_F) = G(F_\infty) \times G(\mathbb{A}_{F,f})$  denote the restricted direct product  $\prod'_v G(F_v)$ , endowed with the usual locally compact topology. Then  $G(F)$  embeds in  $G(\mathbb{A}_F)$  as a discrete subgroup, and if  $Z_n$  denotes the center of  $\text{GL}(n)$ , the homogeneous space  $\text{GL}(n, F)Z(\mathbb{A}_F) \backslash \text{GL}(n, \mathbb{A}_F)$  has finite volume relative to the relatively invariant quotient measure induced by a Haar measure on  $\text{GL}(n, \mathbb{A}_F)$ . An irreducible representation  $\pi$  of  $\text{GL}(n, \mathbb{A}_F)$  is admissible if it admits a factorization as a restricted tensor product  $\otimes'_v \pi_v$ , where each  $\pi_v$  is admissible and for almost all finite places  $v$ ,  $\pi_v$  is *unramified*, i.e., has a no-zero vector fixed by  $K_v = \text{GL}(n, \mathcal{O}_v)$ . (Here, as usual,  $\mathcal{O}_v$  denotes the ring of integers of the local completion  $F_v$  of  $F$  at  $v$ .)

Fixing a unitary idele class character  $\omega$ , which can be viewed as a character of  $Z_n(\mathbb{A}_F)$ , we may consider the space

$$(1.2.1) \quad L^2(n, \omega) := L^2(\mathrm{GL}(n, F)Z(\mathbb{A}_F) \backslash \mathrm{GL}(n, \mathbb{A}_F), \omega),$$

which consists of (classes of) functions on  $\mathrm{GL}(n, \mathbb{A}_F)$  which are left-invariant under  $\mathrm{GL}(n, F)$ , transform under  $Z(\mathbb{A}_F)$  according to  $\omega$ , and are square-integrable modulo  $\mathrm{GL}(n, F)Z(\mathbb{A}_F)$ . Clearly,  $L^2(n, \omega)$  is a unitary representation of  $\mathrm{GL}(n, \mathbb{A}_F)$  under the right translation action on functions. The **space of cusp forms**, denoted  $L_0^2(n, \omega)$ , consists of functions  $\varphi$  in  $L^2(n, \omega)$  which satisfy the following for *every* unipotent radical  $U$  of a standard parabolic subgroup  $P = MU$ :

$$(1.2.2) \quad \int_{U(F) \backslash U(\mathbb{A}_F)} \varphi(ux) = 0.$$

To say that  $P$  is a standard parabolic means that it contains the *Borel subgroup* of upper triangular matrices in  $\mathrm{GL}(n)$ . A basic fact asserts that  $L_0^2(n, \omega)$  is a subspace of the discrete spectrum of  $L^2(n, \omega)$ .

By a **unitary cuspidal** (automorphic) representation  $\pi$  of  $\mathrm{GL}_n(\mathbb{A}_F)$ , we will mean an irreducible, unitary representation occurring in  $L_0^2(n, \omega)$ . We will, by abuse of notation, also denote the underlying admissible representation by  $\pi$ . (To be precise, the unitary representation is on the Hilbert space completion of the admissible space.) Roughly speaking, unitary automorphic representations of  $\mathrm{GL}(n, \mathbb{A}_F)$  are those which appear weakly in  $L^2(n, \omega)$  for some  $\omega$ . We will refrain from recalling the definition precisely, because we will work totally with the subclass of *isobaric automorphic representations*, for which one can take Theorem 1.2.10 (of Langlands) below as their definition.

If  $\pi$  is an admissible representation of  $\mathrm{GL}(n, \mathbb{A}_F)$ , then for any  $z \in \mathbb{C}$ , we define the *analytic Tate twist* of  $\pi$  by  $z$  to be

$$(1.2.3) \quad \pi(z) := \pi \otimes |\cdot|^z,$$

where  $|\cdot|^z$  denotes the 1-dimensional representation of  $\mathrm{GL}(n, \mathbb{A}_F)$  given by

$$g \rightarrow |\det(g)|^z = e^{z \log(|\det(g)|)}.$$

Since the adelic absolute value  $|\cdot|$  takes  $\det(g)$  to a positive real number, its logarithm is well defined.

In general, by a *cuspidal automorphic representation*, we will mean an irreducible admissible representation of  $\mathrm{GL}(n, \mathbb{A}_F)$  for which there exists a real number  $w$ , which we will call the *weight of  $\pi$*  such that the Tate twist

$$(1.2.4) \quad \pi_u := \pi(w/2)$$

is a unitary cuspidal representation. Note that the central character of  $\pi$  and of its unitary *avatar*  $\pi_u$  are related as follows:

$$(1.2.5) \quad \omega_\pi = \omega_{\pi_u} | \cdot |^{-nw/2},$$

which is easily checked by looking at the situation at the unramified primes, which suffices.

For any irreducible, automorphic representation  $\pi$  of  $\mathrm{GL}(n, \mathbb{A}_F)$ , there is an associated  $L$ -function  $L(s, \pi) = L(s, \pi_\infty) L(s, \pi_f)$ , called the *standard  $L$ -function* ([J]) of  $\pi$ . It has an Euler product expansion

$$(1.2.6) \quad L(s, \pi) = \prod_v L(s, \pi_v),$$

convergent in a right-half plane. If  $v$  is an archimedean place, then one knows (cf. [La1]) how to associate a semisimple  $n$ -dimensional  $\mathbb{C}$ -representation  $\sigma(\pi_v)$  of the Weil group  $W_{F_v}$ , and  $L(\pi_v, s)$  identifies with  $L(\sigma_v, s)$ . We will *normalize* this correspondence  $\pi_v \rightarrow \sigma(\pi_v)$  in such a way that it respects algebraicity. Moreover, if  $v$  is a finite place where  $\pi_v$  is unramified, there is a corresponding semisimple conjugacy class  $A_v(\pi)$  (or  $A(\pi_v)$ ) in  $\mathrm{GL}(n, \mathbb{C})$  such that

$$(1.2.7) \quad L(s, \pi_v) = \det(1 - A_v(\pi)T)^{-1} |_{T=q_v^{-s}}.$$

We may find a diagonal representative  $\mathrm{diag}(\alpha_{1,v}(\pi), \dots, \alpha_{n,v}(\pi))$  for  $A_v(\pi)$ , which is unique up to permutation of the diagonal entries. Let  $[\alpha_{1,v}(\pi), \dots, \alpha_{n,v}(\pi)]$  denote the unordered  $n$ -tuple of complex numbers representing  $A_v(\pi)$ . Since  $W_{F,v}^{\mathrm{ab}} \simeq F_v^*$ ,  $A_v(\pi)$  clearly defines an abelian  $n$ -dimensional representation  $\sigma(\pi_v)$  of  $W_{F,v}$ . If  $\underline{1}$  denotes the trivial representation of  $\mathrm{GL}(1, \mathbb{A}_F)$ , which is cuspidal, we have

$$L(s, \underline{1}) = \zeta_F(s),$$

the Dedekind zeta function of  $F$ . (Strictly speaking, we should take  $L(s, \underline{1}_f)$  on the left, since the right hand side is missing the archimedean factor, but this is not serious.)

The fundamental work of Godement and Jacquet, when used in conjunction with the Rankin-Selberg theory (see 1.3 below), yields the following:

**Theorem 1.2.8** ([J]) *Let  $n \geq 1$ , and  $\pi$  a non-trivial cuspidal automorphic representation of  $GL(n, \mathbb{A}_F)$ . Then  $L(s, \pi)$  is entire. Moreover, for any finite set  $S$  of places of  $F$ , the incomplete  $L$ -function*

$$L^S(s, \pi) = \prod_{v \notin S} L(s, \pi_v)$$

*is holomorphic and non-zero in  $\Re(s) > w + 1$  if  $\pi$  has weight  $w$ . Moreover, there is a functional equation*

$$(1.2.9) \quad L(w + 1 - s, \pi^\vee) = \varepsilon(s, \pi) L(s, \pi)$$

*with*

$$\varepsilon(s, \pi) = W(\pi) N_\pi^{(w+1)/2-s}.$$

*Here  $N_\pi$  denotes the norm of the conductor  $\mathcal{N}_\pi$  of  $\pi$ , and  $W(\pi)$  is the root number of  $\pi$ .*

Of course when  $w = 0$ , i.e., when  $\pi$  is unitary, the statement comes to a more familiar form. When  $n = 1$ , a  $\pi$  is simply an idele class character and this result is due to Hecke.

By the theory of Eisenstein series, there is a sum operation  $\boxplus$  ([La2], [JS1]):

**Theorem 1.2.10** ([JS1]) *Given any  $m$ -tuple of cuspidal automorphic representations  $\pi_1, \dots, \pi_m$  of  $GL(n_1, \mathbb{A}_F), \dots, GL(n_m, \mathbb{A}_F)$  respectively, there exists an irreducible, automorphic representation  $\pi_1 \boxplus \dots \boxplus \pi_m$  of  $GL(n, \mathbb{A}_F)$ ,  $n = n_1 + \dots + n_m$ , which is unique up to equivalence, such that for any finite set  $S$  of places,*

$$(1.2.11) \quad L^S(s, \boxplus_{j=1}^m \pi_j) = \prod_{j=1}^m L^S(s, \pi_j).$$

Call such a (Langlands) sum  $\pi \simeq \boxplus_{j=1}^m \pi_j$ , with each  $\pi_j$  cuspidal, an *isobaric automorphic*, or just *isobaric* (if the context is clear), representation. Denote by  $\text{ram}(\pi)$  the finite set of finite places where  $\pi$  is ramified, and let  $\mathfrak{N}(\pi)$  be its conductor.

For every integer  $n \geq 1$ , set:

$$(1.2.12) \quad \mathcal{A}(n, F) = \{\pi : \text{isobaric representation of } \mathrm{GL}(n, \mathbb{A}_F)\} / \simeq,$$

and

$$\mathcal{A}_0(n, F) = \{\pi \in \mathcal{A}(n, F) \mid \pi \text{ cuspidal}\}.$$

Put  $\mathcal{A}(F) = \cup_{n \geq 1} \mathcal{A}(n, F)$  and  $\mathcal{A}_0(F) = \cup_{n \geq 1} \mathcal{A}_0(n, F)$ .

**Remark 1.2.13.** One can also define the analogs of  $\mathcal{A}(n, F)$  for local fields  $F$ , where the “cuspidal” subset  $\mathcal{A}_0(n, F)$  consists of essentially square-integrable representations of  $\mathrm{GL}(n, F)$ . See [La3] (or [Ra1]) for details.

Given any polynomial representation

$$(1.2.14) \quad r : \mathrm{GL}(n, \mathbb{C}) \rightarrow \mathrm{GL}(N, \mathbb{C}),$$

one can associate an  $L$ -function to the pair  $(\pi, r)$ , for any isobaric automorphic representation  $\pi$  of  $\mathrm{GL}(n, \mathbb{A}_F)$ :

$$(1.2.15) \quad L(s, \pi; r) = \prod_v L(s, \pi_v; r),$$

in such a way that at any finite place  $v$  where  $\pi$  is *unramified* with residue field  $\mathbb{F}_q$ ,

$$(1.2.16) \quad L(s, \pi_v; r) = \det(1 - A_v(\pi; r)T)^{-1}|_{T=q_v^{-s}},$$

with

$$(1.2.17) \quad A_v(\pi; r) = r(A_v(\pi)).$$

The conjugacy class  $A_v(\pi; r)$  in  $\mathrm{GL}(N, \mathbb{C})$  is again represented by an unordered  $N$ -tuple of complex numbers.

The **Principle of Functoriality** predicts the existence of an isobaric automorphic representation  $r(\pi)$  of  $\mathrm{GL}(N, \mathbb{A}_F)$  such that

$$(1.2.18) \quad L(s, r(\pi)) = L(s, \pi; r)$$

A weaker form of the conjecture, which suffices for questions like what we are considering, asserts that this identity holds outside a finite set  $S$  of places.

This conjecture is known in the following cases of  $(\mathbf{n}, \mathbf{r})$ :

$$(1.2.19)$$

(**2**,  $\text{sym}^2$ ): Gelbart-Jacquet ([GJ])

(**2**,  $\text{sym}^3$ ): Kim-Shahidi ([KSh1])

(**2**,  $\text{sym}^4$ ): Kim ([K])

(**4**,  $\Lambda^2$ ): Kim ([K])

In this paper we will make use of the last instance of functoriality, namely the *exterior square* transfer of automorphic forms from  $\text{GL}(4)$  to  $\text{GL}(6)$ .

### 1.3 Rankin-Selberg $L$ -functions

The results here are due to the independent and partly complementary, deep works of Jacquet, Piatetski-Shapiro and Shalika, and of Shahidi. Let  $\pi, \pi'$  be isobaric automorphic representations in  $\mathcal{A}(n, F), \mathcal{A}(n', F)$  respectively. Then there exists an associated Euler product  $L(s, \pi \times \pi')$  ([JPSS], [JS1], [Sh1,2], [MW], [CoPS2]), which converges in  $\{\Re(s) > 1\}$ , and admits a meromorphic continuation to the whole  $s$ -plane and satisfies the functional equation, which is given in the unitary case by

$$(1.3.1) \quad L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1-s, \pi^\vee \times \pi'^\vee),$$

with

$$\varepsilon(s, \pi \times \pi') = W(\pi \times \pi') N(\pi \times \pi')^{\frac{1}{2}-s},$$

where the *conductor*  $N(\pi \times \pi')$  is a positive integer not divisible by any rational prime not intersecting the ramification loci of  $F/\mathbb{Q}$ ,  $\pi$  and  $\pi'$ , while  $W(\pi \times \pi')$  is the *root number* in  $\mathbb{C}^*$ . As in the Galois case,  $W(\pi \times \pi') W(\pi^\vee \times \pi'^\vee) = 1$ , so that  $W(\pi \times \pi') = \pm 1$  when  $\pi, \pi'$  are self-dual.

It is easy to deduce the functional equation when  $\pi, \pi'$  are not unitary. If they are cuspidal of weights  $w, w'$  respectively, the functional equation relates  $s$  to  $w + w' + 1 - s$ . Moreover, since  $\pi^\vee, \pi'^\vee$  have respective weights  $-w, -w'$ ,  $\pi \times \pi^\vee$  and  $\pi' \times \pi'^\vee$  still have weight 0.

When  $v$  is archimedean or a finite place unramified for  $\pi, \pi'$ ,

$$(1.3.2) \quad L_v(s, \pi \times \pi') = L(s, \sigma(\pi_v) \otimes \sigma(\pi'_v)).$$

In the archimedean situation,  $\pi_v \rightarrow \sigma(\pi_v)$  is the arrow to the representations of the Weil group  $W_{F_v}$  given by [La1]. When  $v$  is an unramified finite

place,  $\sigma(\pi_v)$  is defined in the obvious way as the sum of one dimensional representations defined by the Langlands class  $A(\pi_v)$ .

When  $n = 1$ ,  $L(s, \pi \times \pi') = L(s, \pi\pi')$ , and when  $n = 2$  and  $F = \mathbb{Q}$ , this function is the usual Rankin-Selberg  $L$ -function, extended to arbitrary global fields by Jacquet.

**Theorem 1.3.3** ([JS1], [JPSS]) *Let  $\pi \in \mathfrak{A}_0(n, F)$ ,  $\pi' \in \mathfrak{A}_0(n', F)$ , and  $S$  a finite set of places. Then  $L^S(s, \pi \times \pi')$  is entire unless  $\pi$  is of the form  $\pi'^\vee \otimes |\cdot|^w$ , in which case it is holomorphic outside  $s = w, 1 - w$ , where it has simple poles.*

The *Principle of Functoriality* implies in this situation that given  $\pi, \pi'$  as above, there exists an isobaric automorphic representation  $\pi \boxtimes \pi'$  of  $\mathrm{GL}(nn', \mathbb{A}_F)$  such that

$$(1.3.4) \quad L(s, \pi \boxtimes \pi') = L(s, \pi \times \pi').$$

The (conjectural) functorial product  $\boxtimes$  is the automorphic analogue of the usual tensor product of Galois representations. For the importance of this product, see [Ra1], for example.

One can always construct  $\pi \boxtimes \pi'$  as an *admissible* representation of  $\mathrm{GL}(nn', \mathbb{A}_f)$ , but the subtlety lies in showing that this product is automorphic.

The automorphy of  $\boxtimes$  is known in the following cases, which will be useful to us:

(1.3.5)

$$(\mathbf{n}, \mathbf{n}') = (\mathbf{2}, \mathbf{2}): \quad ([\mathrm{Ra}2])$$

$$(\mathbf{n}, \mathbf{n}') = (\mathbf{2}, \mathbf{3}): \quad \text{Kim-Shahidi ([KSh1])}$$

The reader is referred to section 11 of [Ra4], which contains some refinements, explanations, refinements and (minor) errata for [Ra2]. It may be worthwhile remarking that Kim and Shahidi use the functorial product on  $\mathrm{GL}(2) \times \mathrm{GL}(3)$  which they construct to prove the *symmetric cube* lifting for  $\mathrm{GL}(2)$  mentioned in the previous section (see (1.2.11)). A *cuspidality criterion* for the image under this transfer is proved in [Ra-W], with an application to the cuspidal cohomology of congruence subgroups of  $\mathrm{SL}(6, \mathbb{Z})$ .

## 1.4 Modularity and the problem at hand

The general Langlands philosophy asserts that if  $\rho_\ell$  is an  $n$ -dimensional  $\ell$ -adic representation of  $\mathcal{G}_{\mathbb{Q}}$  arising as a factor of the cohomology of a smooth, projective variety over  $\mathbb{Q}$ , then there is an isobaric automorphic representation  $\pi$  of  $\mathrm{GL}(n, \mathbb{A})$  such that for a suitable finite set  $S$  of places (including  $\infty$ ), we have an identity of the form

$$(1.4.1) \quad L^S(s, \rho_\ell) = L^S(s, \pi).$$

When this happens, we will say that  $\rho_\ell$  is *modular*, and we will write

$$(1.4.2) \quad \rho_\ell \leftrightarrow \pi.$$

One says that  $\rho_\ell$  is *strongly modular* if the identity (1.4.1) holds for the full  $L$ -function, i.e., with  $S$  empty.

Recall from (the end of) section 1.1 that a striking conjecture of Fontaine and Mazur ([FoM]) asserts that a Galois representation  $\rho_\ell$  comes from arithmetical geometry, as required in the modularity conjecture above, if it is potentially semistable at  $\ell$  and has good reduction at almost all primes.

Special cases of the modularity conjecture were known earlier, the most famous one being the modularity conjecture for the  $\ell$ -adic representations  $\rho_\ell$  defined by the Galois action on the  $\ell$ -power division points of elliptic curves  $E$  over  $\mathbb{Q}$ , proved recently in the spectacular works of Wiles, Taylor, Diamond, Conrad and Breuil.

We will not consider any such (extremely) difficult question in this article. Instead we will be interested in the following:

**Question 1.4.3** *When a modular  $\rho_\ell$  is irreducible, is the corresponding  $\pi$  cuspidal? And conversely?*

This seemingly reasonable question turns out to be hard to check in dimensions  $n > 2$ .

One thing that is clear is that the  $\pi$  associated to any  $\rho_\ell$  needs to be *algebraic* in the sense of Clozel ([Cl1]). To define the notion of *algebraicity*, first recall that by Langlands, the archimedean component  $\pi_\infty$  is associated to an  $n$ -dimensional representation  $\sigma(\pi_\infty)$ , sometimes written  $\sigma_\infty(\pi)$ , of the real Weil group  $W_{\mathbb{R}}$ , with corresponding equality of the archimedean  $L$ -factors  $L_\infty(s, \rho_\ell)$  and  $L(s, \pi_\infty)$ . We will normalize things so that the correspondence



is algebraic. One can explicitly describe  $W_{\mathbb{R}}$  as  $\mathbb{C}^* \cup j\mathbb{C}^*$ , with  $jzj^{-1} = \bar{z}$  and  $j^2 = -1$ . One gets a canonical exact sequence

$$(1.4.4) \quad 1 \rightarrow \mathbb{C}^* \rightarrow W_{\mathbb{R}} \rightarrow \mathcal{G}_{\mathbb{R}} \rightarrow 1$$

which represents the unique non-trivial extension of  $\mathcal{G}_{\mathbb{R}}$  by  $\mathbb{C}^*$ . One has a decomposition

$$(1.4.5) \quad \sigma(\pi_{\infty})|_{\mathbb{C}^*} \simeq \bigoplus_{j=1}^n \xi_j,$$

where each  $\xi_j$  is a (quasi-)character of  $\mathbb{C}^*$ . One says that  $\pi$  is *algebraic* when every one of the characters  $\chi_j$  is algebraic, i.e., there are integers  $p_j, q_j$  such that

$$(1.4.6) \quad \chi_j(z) = z^{p_j} \bar{z}^{q_j}.$$

This is analogous to having a *Hodge structure*, which is what one would expect if  $\pi$  were to be related to a geometric object.

One says that  $\pi$  is *regular* if for all  $i \neq j$ ,  $\chi_i \neq \chi_j$ . In other words, each character  $\chi_j$  appears in the restriction of  $\sigma_{\infty}(\pi)$  to  $\mathbb{C}^*$  with *multiplicity one*. We say (following [BHR]) that  $\pi$  is *semi-regular* if the multiplicity of each  $\chi_j$  is at most 2.

When  $n = 2$ , any  $\pi$  defined by a classical *holomorphic newform*  $f$  of weight  $k \geq 1$  is algebraic and semi-regular. It is regular iff  $k \geq 2$ . One also expects any *Maass waveform*  $\varphi$  of *weight* 0 and *eigenvalue*  $1/4$  for the *hyperbolic Laplacian* to be algebraic; there are interesting examples of this sort coming from the work of Langlands (resp. Tunnell) on tetrahedral (resp. octahedral) Galois representations  $\rho$  which are *even*; the *odd* ones correspond to holomorphic newforms of weight 1. We will not consider the even situation in this article.

Given a holomorphic newform  $f(z) = \sum_{n=1}^{\infty} a_n q^n$ ,  $q = e^{2\pi iz}$ , of weight 2, resp.  $k \geq 3$ , resp.  $k = 1$ , level  $N$  and character  $\omega$ , one knows by Eichler and Shimura, resp. Deligne ([De]), resp. Deligne-Serre ([DeS]), that there is a continuous, irreducible representation

$$(1.4.7) \quad \rho_{\ell} : \mathcal{G}_{\mathbb{Q}} \rightarrow \mathrm{GL}(2, \mathbb{Q}_{\ell})$$

such that for all primes  $p \nmid N\ell$ ,

$$\mathrm{tr}(F_{r_p} | \rho_{\ell}) = a_p$$

and

$$\det(\rho_\ell) = \omega \chi_\ell^{k-1},$$

where  $\chi_\ell$  is the  $\ell$ -adic *cyclotomic character* of  $\mathcal{G}_{\mathbb{Q}}$ , given by the Galois action on the  $\ell$ -power roots of unity in  $\overline{\mathbb{Q}}$ , and  $Fr_p$  is the *geometric Frobenius* at  $p$ , which is the inverse of the arithmetic Frobenius.

## 1.5 Parity

We will first introduce this crucial concept over the base field  $\mathbb{Q}$ , as that is what is needed in the remainder of the article.

We will need to restrict our attention to those isobaric forms  $\pi$  on  $GL(n)/\mathbb{Q}$  which are odd in a suitable sense. It is instructive to first consider the case of a classical holomorphic newform  $f$  of weight  $k \geq 1$  and character  $\omega$  relative to the congruence subgroup  $\Gamma_0(N)$ . Since  $\Gamma_0(N)$  contains  $-I$ , it follows that  $\omega(-1) = (-1)^k$ . One could be tempted to call  $a\pi$  defined by such an  $f$  to be even (or odd) according as  $\omega$  is even (or odd), but it would be a wrong move. One should look not just at  $\omega$ , but at the determinant of the associated  $\rho_\ell$ , i.e., the  $\ell$ -adic character  $\omega \chi_\ell^{k-1}$ , which is odd for all  $k$ ! So all such  $\pi$  defined by holomorphic newforms are *arithmetically* odd. The only even ones for  $GL(2)$  are (analytic Tate twists of) Maass forms of weight 0 and Laplacian eigenvalue  $1/4$ .

The maximal abelian quotient of  $W_{\mathbb{R}}$  is  $\mathbb{R}^*$ , and the restriction of the abelianization map to  $\mathbb{C}^*$  identifies with the norm map  $z \rightarrow |z|$ . So every (quasi)-character  $\xi$  of  $W_{\mathbb{R}}$  identifies with one of  $\mathbb{R}^*$ , given by  $x \rightarrow \text{sgn}(x)^a |x|^t$  for some  $t$ , with  $a \in \{0, 1\}$ . Clearly,  $\xi$  determines, and is determined by  $(t, a)$ . If  $\pi$  is an isobaric automorphic representation, let  $\sigma_\infty[\xi]$  denote, for each such  $\xi$ , the  $\xi$ -isotypic component of  $\sigma_\infty(\pi)$ . The *sign group*  $\mathbb{R}^*/\mathbb{R}_+^*$  acts on each isotypic component. Let  $m_+(\pi, \xi)$  (resp.  $m_-(\pi, \xi)$ ) denote the multiplicity of the eigenvalue  $+1$  (resp.  $-1$ ), under the action of  $\mathbb{R}^*/\mathbb{R}_+^*$  on  $\sigma_\infty(\xi)$ .

**Definition 1.5.1** *Call an isobaric automorphic representation  $\pi$  of  $GL(n, \mathbb{A})$  odd if for every one-dimensional representation  $\xi$  of  $W_{\mathbb{R}}$  occurring in  $\sigma_\infty(\pi)$ ,*

$$|m_+(\pi, \xi) - m_-(\pi, \xi)| \leq 1.$$

Clearly, when the dimension of  $\sigma_\infty[\xi]$  is even, the multiplicity of  $+1$  as an eigenvalue of the sign group needs to be equal to the multiplicity of  $-1$  as an eigenvalue.

Under this definition, all forms on  $GL(1)/\mathbb{Q}$  are odd. So are the  $\pi$  on  $GL(2)/\mathbb{Q}$  which are defined by holomorphic newforms of weight  $k \geq 2$ . The reason is that  $\pi_\infty$  is (for  $k \geq 2$ ) a discrete series representation, and the corresponding  $\sigma_\infty(\pi)$  is an irreducible 2-dimensional representation of  $W_{\mathbb{R}}$  induced by the (quasi)-character  $z \rightarrow z^{-(k-1)}$  of the subgroup  $\mathbb{C}^*$  of index 2, and our condition is vacuous. On the other hand, if  $k = 1$ ,  $\sigma_\infty(\pi)$  is a reducible 2-dimensional representation, given by  $\underline{1} \oplus sgn$ . The eigenvalues are 1 on  $\sigma_\infty(\underline{1})$  and  $-1$  on  $\sigma_\infty(sgn)$ . On the other hand, a Maass form of weight 0 and  $\lambda = 1/4$ , the eigenvalue 1 (or  $-1$ ) occurs with multiplicity 2, making the  $\pi$  it defines an even representation. So our definition is a good one and gives what we know for  $n = 2$ .

For any  $n$ , note that if  $\pi$  is algebraic and regular, it is automatically odd. If  $\pi$  is algebraic and semi-regular, each isotypic space is one or two-dimensional, and in the latter case, we want both eigenvalues to occur for  $\pi$  to be odd.

Finally, if  $F$  is any number field with a real place  $u$ , we can define, in exactly the same way, when an algebraic, isobaric automorphic representation of  $GL(n, \mathbb{A}_F)$  is *arithmetically odd at  $u$* . If  $F$  is *totally real*, then we say that  $\pi$  is *totally odd* if it is so at *every* archimedean place.

## 2 The first step in the proof

Let  $\rho_\ell, \pi$  be as in Theorem A. Since  $\rho_\ell$  is irreducible, it is in particular semisimple. Suppose  $\pi$  is not cuspidal. We will obtain a contradiction.

**Proposition 2.1** *Let  $\rho_\ell, \pi$  be associated, with  $\pi$  algebraic, semi-regular and odd. Suppose we have, for some  $r > 1$ , an isobaric sum decomposition*

$$(2.2) \quad \pi \simeq \boxplus_{j=1}^r \eta_j,$$

*where each  $\eta_j$  a cuspidal automorphic representation of  $GL(n_j, \mathbb{A})$ , with  $n_j \leq 2$  ( $\forall j$ ). Then  $\rho_\ell$  cannot be irreducible.*

**Corollary 2.3** *Theorem A holds when  $\pi$  admits an isobaric sum decomposition such as (2.2) with each  $n_j \leq 2$ . In particular, it holds for  $n \leq 3$ .*

**Proof of Proposition.** The hypothesis that  $\pi$  is algebraic and semi-regular implies easily that each  $\eta_j$  is also algebraic and semi-regular. Let  $J_m$  denote the set of  $j$  where  $n_j = m$ .

First look at any  $j$  in  $J_1$ . Then the corresponding  $\eta_j$  is an idele class character. Its algebraicity implies that, in classical terms, it corresponds to an algebraic Hecke character  $\nu_j$ . By Serre ([Se]), we may attach an abelian  $\ell$ -adic representation  $\nu_{j,\ell}$  of  $\mathcal{G}_{\mathbb{Q}}$  of dimension 1. It follows that for some finite set  $S$  of places containing  $\ell$ ,

$$(2.4) \quad L^S(s, \nu_{j,\ell}) = L^S(s, \eta_j) \quad \text{whenever} \quad n_j = 1.$$

Next consider any  $j$  in  $J_2$ . If  $\sigma(\eta_{j,\infty})$  is irreducible, then a twist of  $\eta_j$  must correspond to a classical holomorphic newform  $f$  of weight  $k \geq 2$ . Moreover, the algebraicity of  $\eta_j$  forces this twist to be algebraic. Hence by Deligne, there is a continuous representation

$$(2.5) \quad \tau_{j,\ell} : \mathcal{G}_{\mathbb{Q}} \rightarrow \mathrm{GL}(2, \overline{\mathbb{Q}_{\ell}}),$$

ramified only at a finite of primes such that at every  $p \neq \ell$  where the representation is unramified,

$$(2.6) \quad \mathrm{tr}(Fr_p | \tau_{j,\ell}) = a_p(\eta_j),$$

and the determinant of  $\tau_{j,\ell}$  corresponds to the central character  $\omega_j$  of  $\eta_j$ . Moreover,  $\tau_{j,\ell}$  is irreducible, which is not crucial to us here.

We also need to consider the situation, for any fixed  $j \in J_2$ , when  $\sigma(\eta_{j,\infty})$  is reducible, say of the form  $\chi_1 \oplus \chi_2$ . Since  $\eta_j$  is cuspidal, by the *archimedean purity* result of Clozel ([Cl1]),  $\chi_1 \chi_2^{-1}$  must be 1 or *sgn*. The former cannot happen due to the oddness of  $\pi$ . It follows that  $\eta_j$  is defined by a classical holomorphic newform  $f$  of weight 1, and by a result of Deligne and Serre ([DeS]), there is a 2-dimensional  $\ell$ -adic representation  $\tau_{j,\ell}$  of  $\mathcal{G}_{\mathbb{Q}}$  with finite image, which is irreducible, such that (2.6) holds.

Since the set of Frobenius classes  $Fr_p$ , as  $p$  runs over primes outside  $S$ , is dense in the Galois group by Tchebotarev, we must have, by putting all these cases together,

$$(2.7) \quad \rho_{\ell} \simeq (\oplus_{j \in J_1} \nu_{j,\ell}) \oplus (\oplus_{j \in J_2} \tau_{j,\ell}),$$

which contradicts the irreducibility of  $\rho_{\ell}$ , since by hypothesis,  $r = |J_1| + |J_2| \geq 2$ .

□

### 3 The second step in the proof

Let  $\rho_\ell, \pi$  be as in Theorem A. Suppose  $\pi$  is not cuspidal. In view of Proposition 2.1, we need only consider the situation where  $\pi$  is an isobaric sum  $\boxplus_j \eta_j$ , with an  $\eta_j$  being a cusp form on  $GL(m)/\mathbb{Q}$  for some  $m \geq 3$ .

**Proposition 3.1** *Let  $\rho_\ell, \pi$  be associated, with  $\pi$  an algebraic cusp form on  $GL(n)/\mathbb{Q}$  which is semi-regular and odd. Suppose we have an isobaric sum decomposition*

$$(3.2) \quad \pi \simeq \eta \boxplus \eta',$$

where  $\eta$  is a cusp form on  $GL(3)/\mathbb{Q}$  and  $\eta'$  an isobaric automorphic representation of  $GL(r, \mathbb{A})$  for some  $r \geq 1$ . Moreover, assume that there is an  $r$ -dimensional  $\ell$ -adic representation  $\tau'_\ell$  of  $\mathcal{G}_\mathbb{Q}$  associated to  $\eta'$ . Then we have the isomorphism of  $\mathcal{G}_\mathbb{Q}$ -modules:

$$(3.3) \quad \rho_\ell^\vee \oplus (\rho_\ell \otimes \tau'_\ell) \simeq \Lambda^2(\rho_\ell) \oplus \tau'^\vee_\ell \oplus \text{sym}^2(\tau'_\ell).$$

**Corollary 3.4** *Let  $\rho_\ell, \pi$  be associated, with  $\pi$  algebraic, semi-regular and odd. Suppose  $\pi$  admits an isobaric sum decomposition such as (3.2) with  $r \leq 2$ . Then  $\rho_\ell$  is reducible.*

**Proposition 3.1**  $\implies$  **Corollary 3.4**: When  $r \leq 2$ ,  $\eta'$  is either an isobaric sum of algebraic Hecke characters or cuspidal, in which case, thanks to the oddness, it is defined by a classical cusp form on  $GL(r)/\mathbb{Q}$  of weight  $\geq 1$ . In either case we have, as seen in the previous section, the existence of the associated  $\ell$ -adic representation  $\tau'_\ell$ , which is irreducible exactly when  $\eta_\ell$  is cuspidal. Then by the Proposition, the decomposition (3.3) holds. If  $r = 1$  or  $r = 2$  with  $\eta'$  Eisensteinian, (3.3) implies that a 1-dimensional representation (occurring in  $\tau'_\ell$ ) is a summand of a twist of either  $\rho_\ell$  or  $\rho_\ell^\vee$ . Hence the Corollary. □

Combining Corollary 2.3 and Corollary 3.4, we see that the irreducibility of  $\rho_\ell$  forces the corresponding  $\pi$  to be cuspidal when  $n \leq 4$  under the hypotheses of Theorem A. So we obtain the following:

**Corollary 3.5** *Theorem A holds for  $n \leq 4$ .*

**Proof of Proposition 3.1.** By hypothesis, we have a decomposition as in (3.2), and an  $\ell$ -adic representation  $\tau'_\ell$  associated to  $\eta'$ .

As a short digression let us note that if  $\eta$  were essentially self-dual and regular, we could exploit its algebraicity, and by appealing to [Pic] associate a 3-dimensional  $\ell$ -adic representation to  $\eta$ . The Proposition 3.1 will follow in that case, as in the proof of Proposition 2.1. However, we cannot (and do not wish to) assume either that  $\eta$  is essentially self-dual or that it is regular. We have to appeal to another idea, and here it is.

Let  $S$  be a finite set of primes including the archimedean and ramified ones. At any  $p$  outside  $S$ , let  $\pi_p$  be represented by an unordered  $(3+r)$ -tuple  $\{\alpha_1, \dots, \alpha_{3+r}\}$  of complex numbers, and we may assume that  $\eta_p$  (resp.  $\eta'_p$ ) is represented by  $\{\alpha_1, \alpha_2, \alpha_3\}$  (resp.  $\{\alpha_4, \dots, \alpha_{3+r}\}$ ). It is then straightforward to check that

$$(3.6) \quad L(s, \pi_p; \Lambda^2) = L(s, \eta_p^\vee) L(s, \eta_p \times \eta'_p) L(s, \eta'; \Lambda^2).$$

One can also deduce this as follows. Let  $\sigma(\beta)$  denote, for any irreducible admissible representation  $\beta$  of  $\mathrm{GL}(m, \mathbb{Q}_p)$ , the  $m$ -dimensional representation of the extended Weil group  $W'_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \mathbb{C})$  defined by the local Langlands correspondence (cf. [HaT], [He]). For any representation  $\sigma$  of  $W'_{\mathbb{Q}_p}$  which splits as a direct sum  $\tau \oplus \tau'$ , we have

$$(3.7) \quad \Lambda^2(\sigma) \simeq \Lambda^2(\tau) \oplus \tau \otimes \tau' \oplus \Lambda^2(\tau'),$$

with  $\Lambda^2(\tau') = 0$  if  $\tau'$  is 1-dimensional and

$$(3.8) \quad \Lambda^2(\tau) \simeq \tau^\vee \quad \text{when} \quad \dim(\tau) = 3.$$

In fact, this shows that the identity (3.6) works at the ramified primes as well, but we do not need it.

Now, since  $\pi^\vee \simeq \eta^\vee \boxplus \eta'^\vee$ , we get by putting (3.2) and (3.6) together,

$$(3.9) \quad L(s, \pi_p^\vee) L(s, \pi_p \times \eta'_p) L(s, \eta'_p; \Lambda^2) = L(s, \pi_p; \Lambda^2) L(s, \eta_p'^\vee) L(s, \eta'_p \times \eta'_p).$$

Appealing to Tchebotarev, and using the correspondences  $\pi \leftrightarrow \rho_\ell$  and  $\eta' \leftrightarrow \tau'_\ell$ , we obtain the following isomorphism of  $\mathcal{G}_{\mathbb{Q}}$ -representations:

$$(3.10) \quad \rho_\ell^\vee \oplus (\rho_\ell \otimes \tau'_\ell) \oplus \Lambda^2(\tau'_\ell) \simeq \Lambda^2(\rho_\ell) \oplus \tau_\ell'^\vee \oplus (\tau'_\ell \otimes \tau'_\ell).$$

Using the decomposition

$$\tau'_\ell \otimes \tau'_\ell \simeq \text{sym}^2(\tau'_\ell) \oplus \Lambda^2(\tau'_\ell),$$

we then obtain (3.3) from (3.10). □

## 4 Galois representations attached to regular, selfdual cusp forms on $\text{GL}(4)$

A cusp form  $\Pi$  on  $\text{GL}(m)/F$ ,  $F$  a number field, is said to be *essentially selfdual* iff  $\Pi^\vee \simeq \Pi \otimes \lambda$  for an idele class character  $\lambda$ ; it is *selfdual* if  $\lambda = 1$ . We will call such a  $\lambda$  a *polarization*. Let us call  $\Pi$  *almost selfdual* if there is a polarization  $\lambda$  of the form  $\mu^2|\cdot|^t$  for some  $t \in \mathbb{C}$  and a finite order character  $\mu$ ; in this case, one sees that  $(\Pi \otimes \mu)^\vee \simeq \Pi \otimes \mu|\cdot|^t$ , or equivalently,  $\Pi \otimes \mu|\cdot|^{t/2}$  is selfdual. Clearly, if  $\Pi$  is essentially selfdual, then it becomes, under base change ([AC]), almost selfdual over a finite cyclic extension  $K$  of  $F$ .

Note that when  $\Pi$  is essentially selfdual relative to  $\lambda$ , it is immediate that  $\lambda_\infty$  occurs in the isobaric sum decomposition of  $\Pi_\infty \boxtimes \Pi_\infty$ , or equivalently,  $\sigma(\lambda_\infty)$  is a constituent of  $\sigma(\Pi_\infty)^{\otimes 2}$ . This implies that if  $\Pi$  is algebraic, then so is  $\Lambda$ , and thus corresponds to an  $\ell$ -adic character  $\lambda_\ell$  of  $\mathcal{G}_\mathbb{Q}$ .

Whether or not  $\Pi$  is algebraic, we have, for any  $S$ ,

$$(4.1) \quad L^S(s, \Pi \times \Pi \otimes \lambda^{-1}) = L^S(s, \Pi, \text{sym}^2 \otimes \lambda^{-1}) L^S(s, \Pi; \Lambda^2 \otimes \lambda^{-1}),$$

The  $L$ -function on the left has a pole at  $s = 1$ , since  $\Pi^\vee \simeq \Pi \otimes \lambda$  by hypothesis. Also, neither of the  $L$ -functions on the right is zero at  $s = 1$  ([JS2]). Consequently, exactly one of the  $L$ -functions on the right of (4.1) admits a pole at  $s = 1$ . One says that  $\Pi$  is of *orthogonal type* ([Ra3]), resp. *symplectic type*, if  $L^S(s, \Pi, \text{sym}^2 \otimes \lambda^{-1})$ , resp.  $L^S(s, \Pi, \Lambda^2 \otimes \lambda^{-1})$  admits a pole at  $s = 1$ .

The following result is a consequence of a synthesis of the results of a number of mathematicians, and it will be crucial to us in the next section, while proving Theorem A for  $n = 5$ .

**Theorem B** *Let  $\Pi$  be a regular, algebraic cusp form on  $GL(4)/\mathbb{Q}$ , which is almost selfdual. Then there exists a continuous representation*

$$R_\ell : \mathcal{G}_\mathbb{Q} \rightarrow GL(4, \overline{\mathbb{Q}}_\ell),$$

*such that*

$$L^S(s, \Pi; \Lambda^2) = L^S(s, \Lambda^2(R_\ell)),$$

*for a finite set  $S$  of primes containing the ramified ones. Moreover, if  $\Pi$  is of orthogonal type, we can show that  $R_\ell$  and  $\Pi$  are associated, i.e., have the same degree 4  $L$ -functions (outside  $S$ ).*

When  $\Pi$  admits a discrete series component  $\Pi_p$  at some (finite) prime  $p$ , a stronger form of this result, and in fact its generalization to  $GL(n)/\mathbb{Q}$ , is due to Clozel ([Cl2]). But in the application considered in the next section, we will not be able to satisfy such a ramification assumption at a finite place.

In the *orthogonal case*,  $\Pi$  descends by the work of Ginzburg-Rallis-Soudry (cf. [So]) to define a regular cusp form  $\beta$  on the split  $SGO(4)/\mathbb{Q}$ , which is given by a pair  $(\pi_1, \pi_2)$  of regular cusp forms on  $GL(2)/\mathbb{Q}$ . By Deligne, there are 2-dimensional (irreducible)  $\ell$ -adic representations  $\tau_{1,\ell}, \tau_{2,\ell}$ , with  $\tau_{j,\ell} \leftrightarrow \pi_j$ ,  $j = 1, 2$ . This leads to the desired 4-dimensional  $\overline{\mathbb{Q}}_\ell$ -representation  $R_\ell := \tau_{1,\ell} \otimes \tau_{2,\ell}$  of  $\mathcal{G}_\mathbb{Q}$  associated to  $\Pi$ , such that

$$(4.2) \quad L^S(s, R_\ell) = L^S(s, \Pi).$$

It may be useful to notice that since the polarization is a square (under the almost selfduality assumption), the associated Galois representation takes values in  $SGO(4, \overline{\mathbb{Q}}_\ell)$ , which is the connected component of  $GO(4, \overline{\mathbb{Q}}_\ell)$ , with quotient  $\{\pm 1\}$ . In the general case, not needed for this article,  $R_\ell$  will need to be either of the type above or of *Asai type* (see [Ra4]), associated to a 2-dimensional  $\overline{\mathbb{Q}}_\ell$ -representation  $\text{Gal}(\overline{\mathbb{Q}}/K)$  for a quadratic extension  $K/\mathbb{Q}$ .

In the (more subtle) *symplectic case*, this Theorem is proved in my joint work [Ra-Sh] with F. Shahidi. We will start with a historical comment and then sketch the proof (for the benefit of the reader). Some years ago, Jacquet, Piatetski-Shapiro and Shalika announced a theorem, asserting that one could descend any  $\Pi$  (of symplectic type on  $GL(4)/\mathbb{Q}$ ) to a generic cusp form  $\beta$  on  $GSp(4)/\mathbb{Q}$  with the same (incomplete) degree 4  $L$ -functions. Unfortunately, this work was never published, except for a part of it in [JSh2]. In [Ra-Sh], Shahidi and I provide an alternate, somewhat more circuitous route, yielding something slightly weaker, but sufficient for many purposes. Here is the idea.



We begin by considering the twist  $\Pi_0 := \Pi \otimes \mu|\cdot|^{t/2}$  instead of  $\Pi$ , to make the polarization is trivial, i.e., so that  $\Pi_0$  has parameter in  $\mathrm{Sp}(4, \mathbb{C})$ . Using the *backwards lifting* results of [GRS] (see also [So]), we get a generic cusp form  $\Pi'$  on the split  $\mathrm{SO}(5)/\mathbb{Q}$ , such that  $\Pi_0 \rightarrow \Pi'$  is functorial at the archimedean and unramified places. Using the isomorphism of  $\mathrm{PSp}(4)/\mathbb{Q}$  with  $\mathrm{SO}(5)/\mathbb{Q}$ , we may lift  $\Pi'$  to a generic cusp form  $\tilde{\Pi}'$  on  $\mathrm{Sp}(4)/\mathbb{Q}$ . By a suitable extension followed by induction, we can associate a generic cusp form  $\Pi_1$  on  $\mathrm{GSp}(4)/\mathbb{Q}$ , such that the following hold:

(4.3)

- (i) The archimedean parameter of  $\Pi_2 := \Pi_1 \otimes \mu^{-1} \cdot |\cdot|^{-t/2}$  is algebraic and regular; and
- (ii)  $L(s, \Pi_{2,p}; \Lambda^2) = L(s, \Pi_p; \Lambda^2)$  at any prime  $p$  where  $\Pi$  is unramified.

We in fact deduce a stronger statement in [Ra-Sh], involving also the ramified primes, but it is not necessary for the application considered in this paper. To continue, part (i) of (4.3) implies that  $\Pi_2$  contributes to the (intersection) cohomology of (the Baily-Borel-Satake compactification over  $\mathbb{Q}$  of) the 3-dimensional Shimura variety  $Sh_K/\mathbb{Q}$  associated to  $\mathrm{GSp}(4)/\mathbb{Q}$ , relative to a compact open subgroup  $K$  of  $\mathrm{GSp}(4, \mathbb{A}_f)$ ;  $Sh_K$  parametrizes principally polarized abelian surfaces with level  $K$ -structure. Now by appealing to the deep (independent) works of G. Laumon ([Lau1,2]) and R. Weissauer ([Wei]), one gets a continuous 4-dimensional  $\ell$ -adic representation  $R_\ell$  of  $\mathcal{G}_\mathbb{Q}$  such that

$$(4.4) \quad L^S(s, \Pi_2) = L^S(s, R_\ell).$$

The assertion of Theorem B now follows by combining (4.3)(ii) and (4.4).  $\square$

## 5 Two useful Lemmas on cusp forms on $\mathrm{GL}(4)$

Let  $F$  be a number field and  $\eta$  a cuspidal automorphic representation of  $\mathrm{GL}(4, \mathbb{A}_F)$ , where  $\mathbb{A}_F := \mathbb{A} \otimes_\mathbb{Q} F$  is the Adele ring of  $F$ . Denote by  $\omega_\eta$  the central character of  $\eta$ .

First let us recall (see (1.2.11)) that by a difficult *theorem of H. Kim* ([K]), there is an isobaric automorphic form  $\Lambda^2(\eta)$  on  $\mathrm{GL}(6)/\mathbb{Q}$  such that

$$(5.1) \quad L(s, \Lambda^2(\eta)) = L(s, \eta; \Lambda^2).$$

**Lemma 5.2**  $\Lambda^2(\eta)$  is essentially selfdual. In fact

$$(5.3) \quad \Lambda^2(\eta)^\vee \simeq \Lambda^2(\eta) \otimes \omega_\eta^{-1}$$

**Proof.** Thanks to the strong multiplicity one theorem for isobaric automorphic representations ([JS1]), it suffices to check this at the primes  $p$  where  $\eta$

is unramified. Fix any such  $p$ , and represent the semisimple conjugacy class  $A_p(\eta)$  by  $[a, b, c, d]$ . Then it is easy to check that

$$(5.4) \quad A_p(\Lambda^2(\eta)) = \Lambda^2(A_p(\eta)) = [ab, ac, ad, bc, bd, cd].$$

Since for any automorphic representation  $\Pi$ , the unordered tuple representing  $A_p(\Pi^\vee)$  consists of the inverses of the elements of tuple representing  $A_p(\pi)$ , and since  $A_p(\omega_\eta) = [abcd]$ , we have

$$(5.5) \quad A_p(\Lambda^2(\eta)^\vee \otimes \omega_\eta) = [(ab)^{-1}, (ac)^{-1}, (ad)^{-1}, (bc)^{-1}, (bd)^{-1}, (cd)^{-1}] \otimes [abcd],$$

which is none other than  $A_p(\Lambda^2(\eta))$ . The isomorphism (5.3) follows.  $\square$

**Lemma 5.6** *Let  $\eta$  be a cusp form on  $GL(4)/F$  with trivial central character. Suppose  $\eta^\vee \not\cong \eta$ . Then there are infinitely many primes  $P$  in  $\mathcal{O}_F$  where  $\eta_P$  is unramified such that 1 is not an eigenvalue of the conjugacy class  $A_P(\Lambda^2(\eta))$  of  $\Lambda^2(\eta_P)$ .*

**Proof of Lemma 5.6.** Since  $\eta^\vee \not\cong \eta$ , there exist, by the strong multiplicity one theorem, infinitely many unramified primes  $P$  where  $\eta_P^\vee \not\cong \eta_P$ . Pick any such  $P$ , write

$$A_P(\eta) = [a, b, c, d], \quad \text{with } abcd = 1.$$

The fact that  $\eta_P^\vee \not\cong \eta_P$  implies that the set  $\{a, b, c, d\}$  is not stable under inversion. Hence one of its elements, which we may assume to be  $a$  after renaming, satisfies the following:

$$a \notin \{a^{-1}, b^{-1}, c^{-1}, d^{-1}\}.$$

Equivalently,

$$1 \notin \{a^2, ab, ac, ad\}.$$

On the other hand, we have (5.4), using which we conclude that the only way 1 can be in this set (attached to  $\Lambda^2(\eta_P)$ ) is to have either  $bc$  or  $bd$  or  $cd$  to be 1. But if  $bc = 1$  (resp.  $bd = 1$ ), since  $abcd = 1$ , we must have  $ad = 1$  (resp.  $ac = 1$ ), which is impossible. Similarly, if  $cd = 1$ , we are forced to have  $ab = 1$ , which is also impossible.  $\square$

## 6 Finale

Let  $\rho_\ell, \pi$  be as in Theorem A. In view of Corollary 3.5, we may assume from henceforth that  $n = 5$ , and that  $\pi$  is algebraic and regular. Suppose  $\pi$  is not cuspidal. In view of Corollary 2.3 and Corollary 3.4, we must then have the decomposition

$$(6.1) \quad \pi \simeq \eta \boxplus \nu,$$

where  $\eta$  is an algebraic, regular cusp form on  $\mathrm{GL}(4)/\mathbb{Q}$  and  $\nu$  an algebraic Hecke character, with associated  $\ell$ -adic character  $\nu_\ell$ .

Note that Theorem A needs to be proved under *either* of two hypotheses. To simplify matters a bit, we will make use of the following:

**Lemma 6.2** *There is a character  $\nu_0$  with  $\nu_0^2 = 1$  such that for  $\mu = \nu_0 \nu^{-1}$ , if  $\pi$  is almost selfdual, then so is  $\pi \otimes \mu^{-1}$ .*

**Proof of Lemma** When  $\pi$  is almost selfdual, there exists, by definition, an idele class character  $\mu$  such that  $\pi \otimes \mu$  is selfdual. But this implies, thanks to (6.1) and the cuspidality of  $\eta$ , that  $\eta \otimes \mu$  is selfdual and  $\mu\nu$  is 1 or quadratic. We are done by taking  $\nu_0 = \mu\nu$ . □

Consequently, we may, and we will, replace  $\pi$  by  $\pi \otimes \mu$ ,  $\rho_\ell$  by  $\rho_\ell \otimes \mu_\ell$ ,  $\eta$  by  $\eta \otimes \mu$  and  $\nu$  by  $\nu\mu$ , without jeopardizing the nature of either of the hypotheses of Theorem A. In fact, the *first hypothesis simplifies to assuming that  $\pi$  is selfdual*. Moreover,

$$(6.3) \quad \nu^2 = 1.$$

**Proof of Theorem A when  $\pi$  is almost selfdual:**

We have to rule out the decomposition (6.1), which gives (for any finite set  $S$  of places containing the ramified and unramified ones):

$$(6.4) \quad L^S(s, \pi) = L^S(s, \eta)L^S(s, \nu)$$

As noted above, we may in fact assume that  $\pi$  is selfdual and that  $\nu^2 = 1$ . Then the cusp form  $\eta$  will also be selfdual and algebraic. We may then apply Theorem B and conclude the existence of a 4-dimensional, semisimple  $\ell$ -adic

representation  $\tau_\ell$  associated to  $\eta$ . Then, expanding  $S$  to include  $\ell$ , we see that (6.2) implies, in conjunction with the associations  $\pi \leftrightarrow \rho_\ell$ ,  $\eta \leftrightarrow \tau_\ell$ ,

$$(6.5) \quad L^S(s, \rho_\ell) = L^S(s, \tau_\ell) L^S(s, \nu_\ell).$$

By Tchebotarev, this gives the isomorphism

$$(6.6) \quad \rho_\ell \simeq \tau_\ell \oplus \nu_\ell,$$

which contradicts the irreducibility of  $\rho_\ell$ . □

**Proof of Theorem A for general regular  $\pi$ :**

Suppose we have the decomposition (6.1). Again, we may assume that  $\nu^2 = 1$ .

Let  $\omega = \omega_\pi$  denote the central character of  $\pi$ . Then from (6.1) we obtain

$$(6.7) \quad \omega = \omega_\eta \nu.$$

**Proposition 6.8** *Assume the decomposition (6.1), and denote by  $\omega$  the central character of  $\pi$  with corresponding  $\ell$ -adic character  $\omega_\ell$ .*

(a) *We have the identity*

$$L^S(s, \pi; \Lambda^2) L^S(s, \pi^\vee \otimes \omega \nu) \zeta^S(s) = L^S(s, \pi^\vee; \Lambda^2 \otimes \omega) L^S(\pi \otimes \nu) L^S(s, \omega \nu).$$

(b) *There is an isomorphism of  $\mathcal{G}_\mathbb{Q}$ -modules*

$$\Lambda^2(\rho_\ell) \oplus (\rho_\ell^\vee \otimes \omega_\ell) \oplus \mathbf{1} \simeq (\Lambda^2(\rho_\ell^\vee) \otimes \omega_\ell \nu_\ell) \oplus (\rho_\ell \otimes \nu_\ell) \oplus \omega_\ell \nu_\ell.$$

**Proof of Proposition 6.8.** (a) It is immediate, by checking at each unramified prime, that

$$(6.9) \quad L^S(s, \pi; \Lambda^2) = L^S(s, \Lambda^2(\eta)) L^S(\eta \otimes \nu),$$

and (since  $\nu = \nu^{-1}$ )

$$(6.10) \quad L^S(s, \pi^\vee; \Lambda^2) = L^S(s, \Lambda^2(\eta^\vee)) L^S(\eta^\vee \otimes \nu).$$

Since  $\omega = \omega_\eta \nu$ , we get from Lemma 5.2 that  $\Lambda^2(\eta^\vee)$  is isomorphic to  $\Lambda^2(\eta) \otimes \omega^{-1} \nu$ . Twisting (6.10) by  $\omega \nu$ , and using the fact that

$$(6.11) \quad L^S(\eta^\vee \otimes \omega) = L^S(s, \pi^\vee \otimes \omega) / L^S(s, \omega \nu),$$

we obtain

$$(6.12) \quad L^S(s, \pi^\vee; \Lambda^2 \otimes \omega \nu) L^S(s, \omega \nu) = L^S(s, \Lambda^2(\eta)) L^S(s, \pi^\vee \otimes \omega).$$

Similarly, using (6.9) and the fact that

$$L^S(s, \eta \otimes \nu) = L^S(s, \pi \otimes \nu) / \zeta^S(s),$$

we obtain the identity

$$(6.13) \quad L^S(s, \pi; \Lambda^2) \zeta^S(s) = L^S(s, \Lambda^2(\eta)) L^S(s, \pi \otimes \nu).$$

The assertion of part (a) of the Proposition now follows by comparing (6.12) and (6.13).

(b) Follows from part (a) by applying Tchebotarev, since  $\rho_\ell \leftrightarrow \pi$ . □

**Proposition 6.14** We have

$$(a) \quad \omega \nu = 1.$$

$$(b) \quad \rho_\ell^\vee \simeq \rho_\ell.$$

**Proof of Proposition 6.14.** (a) Since  $\rho_\ell$  is irreducible of dimension 5, it cannot admit a one-dimensional summand, and hence part (b) of Proposition 6.8 implies that either  $\omega_\ell \nu_\ell = 1$  or

$$\omega_\ell \nu_\ell \subset \Lambda^2(\rho_\ell).$$

Since the first case gives the assertion, let us assume that we are in the second case. But then, again since  $\rho_\ell$  is irreducible, and since  $\Lambda^2(\rho_\ell)$  is a summand of  $\rho_\ell \otimes \rho_\ell$ , we must have

$$\rho_\ell^\vee \simeq \rho_\ell \otimes (\omega_\ell \nu_\ell)^{-1}.$$

In other words,  $\rho_\ell$  is essentially selfdual in this case. Then so is  $\pi$ . More explicitly, we have (since  $\nu^2 = 1$ )

$$\eta^\vee \boxplus \nu \simeq \pi^\vee \simeq \pi \otimes \omega^{-1} \nu \simeq \eta \otimes \omega^{-1} \nu \boxplus \omega^{-1}.$$

As  $\eta$  is cuspidal, this forces the identity

$$\nu = \omega^{-1}.$$

Done.

(b) Thanks to part (a) (of this Proposition), we may rewrite part (b) of Proposition 6.8 as giving the isomorphism of  $\mathcal{G}_\mathbb{Q}$ -modules

$$(6.15) \quad \Lambda^2(\rho_\ell) \oplus (\rho_\ell^\vee \otimes \omega_\ell) \simeq (\Lambda^2(\rho_\ell^\vee)) \oplus (\rho_\ell \otimes \nu_\ell).$$

We now need the following:

**Lemma 6.16** *Suppose  $\rho_\ell$  is not selfdual. Then*

$$(6.17) \quad \rho_\ell \otimes \nu_\ell \not\subset \Lambda^2(\rho_\ell).$$

**Proof of Lemma 6.16.** The hypothesis on  $\rho_\ell$  implies that  $\pi$  is not selfdual, and since  $\pi = \eta \boxplus \nu$ ,  $\eta$  is not selfdual either.

Suppose (6.17) is false.. Then, since  $\rho_\ell \leftrightarrow \pi$ , we must have

$$(6.18) \quad A_p(\pi \otimes \nu) \subset A_p(\pi; \Lambda^2), \quad \forall p \notin S,$$

for a *finite* set  $S$  of primes. But we also have

$$(6.19) \quad A_p(\pi \otimes \nu) = A_p(\eta \otimes \nu) \oplus \underline{1}$$

and

$$(6.20) \quad A_p(\pi; \Lambda^2) = A_p(\Lambda^2(\eta)) \oplus A_p(\eta \otimes \nu).$$

Substituting (6.19) and (6.20) in (6.18), we obtain

$$(6.21) \quad \underline{1} \subset A_p(\Lambda^2(\eta)) \quad \forall p \notin S.$$

On the other hand, since  $\eta$  is a non-selfdual cusp form on  $\mathrm{GL}(4)/\mathbb{Q}$  of trivial central character, we may apply Lemma 5.6 with  $F = \mathbb{Q}$ , and conclude that there is an *infinite set of primes*  $T$  such that

$$(6.22) \quad \underline{1} \not\subset A_p(\Lambda^2(\eta)) \quad \forall p \in T,$$

which contradicts (6.21), proving the Lemma. □

In view of the identity (6.15) and Lemma 6.16, we have now proved all of Proposition 6.14. □

We are also done with the proof of Theorem A because  $\pi$  is selfdual when the decomposition (6.1) holds, thanks to the irreducibility of  $\rho_\ell$ , and the selfdual case has already been established (using the algebraic regularity of  $\pi$ ). □

## Bibliography

- [AC] J. Arthur and L. Clozel, *Simple Algebras, Base Change and the Advanced Theory of the Trace Formula*, Ann. Math. Studies **120**, Princeton, NJ (1989).
- [BHR] D. Blasius, M. Harris and D. Ramakrishnan, *Coherent cohomology, limits of discrete series, and Galois conjugation*, Duke Math. Journal **73** (1994), no. 3, 647–685.
- [Cl1] L. Clozel, *Motifs et formes automorphes*, in *Automorphic Forms, Shimura varieties, and L-functions*, vol. I, 77–159, Perspectives in Math. **10** (1990).
- [Cl2] L. Clozel, *Représentations galoisiennes associées aux représentations automorphes autoduales de  $GL(n)$* , Publ. Math. IHES **73**, 97–145 (1991).
- [CoPS1] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for  $GL_n$  II*, J. Reine Angew. Math. **507**, 165–188 (1999).
- [CoPS2] J. Cogdell and I. Piatetski-Shapiro, *Remarks on Rankin-Selberg convolutions*, in *Contributions to automorphic forms, geometry, and number theory*, 255–278, Johns Hopkins Univ. Press, Baltimore, MD (2004).
- [De] P. Deligne, *Formes modulaires et Représentations  $\ell$ -adiques*, (1972).



- [De-S] P. Deligne and J.-P. Serre, *Formes modulaires de poids 1*, Ann. Sci. cole Norm. Sup. (4) **7**, 507–530 (1975).
- [FoM] J.-M. Fontaine and B. Mazur, *Geometric Galois representations*, in *Elliptic curves, modular forms, and Fermat’s last theorem*, 41–78, Ser. Number Theory, I, Internat. Press, Cambridge, MA (1995).
- [GJ] S. Gelbart and H. Jacquet, *A relation between automorphic representations of  $GL(2)$  and  $GL(3)$* , Ann. Scient. Éc. Norm. Sup. (4) **11** (1979), 471–542.
- [GRS] D. Ginzburg, S. Rallis and D. Soudry, *On explicit lifts of cusp forms from  $GL_m$  to classical groups*, Ann. of Math. (2) **150**, no. 3, 807–866 (1999).
- [HaT] M. Harris and R. Taylor, *On the geometry and cohomology of some simple Shimura varieties*, with an appendix by V.G. Berkovich, Annals of Math Studies **151**, Princeton University Press, Princeton, NJ (2001)
- [He] G. Henniart, *Une preuve simple des conjectures de Langlands pour  $GL(n)$  sur un corps  $p$ -adique*, Invent. Math. **139**, no. 2, 439–455 (2000).
- [J] H. Jacquet, *Principal  $L$ -functions of the linear group*, in *Automorphic forms, representations and  $L$ -functions*, Proc. Sympos. Pure Math. **33**, Part 2, 63–86, Amer. Math. Soc., Providence, R.I. (1979).
- [JPSS] H. Jacquet, I. Piatetski-Shapiro and J.A. Shalika, *Rankin-Selberg convolutions*, Amer. J of Math. **105** (1983), 367–464.
- [JS1] H. Jacquet and J.A. Shalika, *Euler products and the classification of automorphic forms I & II*, Amer. J of Math. **103** (1981), 499–558 & 777–815.
- [JS2] H. Jacquet and J.A. Shalika, *Exterior square  $L$ -functions*, in *Automorphic forms, Shimura varieties, and  $L$ -functions*, Vol. II, 143–226, Perspectives in Math. **11** (1990), Academic Press, Boston, MA.
- [K] H. Kim, *Functoriality of the exterior square of  $GL_4$  and the symmetric fourth of  $GL_2$* , with Appendix 1 by D. Ramakrishnan and Appendix 2 by Kim and P. Sarnak, J. Amer. Math. Soc. **16**, no. 1, 139–183 (2003).

- [KSh1] H. Kim and F. Shahidi, *Functorial products for  $GL(2) \times GL(3)$  and the symmetric cube for  $GL(2)$* , With an appendix by Colin J. Bushnell and Guy Henniart, *Annals of Math.* (2) **155**, no. 3, 837–893 (2002).
- [KSh2] H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*, *Duke Math. J.* **112**, no. 1, 177–197 (2002).
- [La1] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*, in *Representation theory and harmonic analysis on semisimple Lie groups*, 101–170, *Math. Surveys Monographs* **31**, AMS, Providence, RI (1989).
- [La2] R.P. Langlands, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, *Proc. symp. Pure Math* **33**, ed. by A. Borel and W. Casselman, part 2, 205–246, *Amer. Math. Soc.*, Providence (1979).
- [La3] R.P. Langlands, *On the notion of an automorphic representation*, *Proc. symp. Pure Math* **33**, ed. by A. Borel and W. Casselman, part 2, 189–217, *Amer. Math. Soc.*, Providence (1979).
- [Lau1] G. Laumon, *Sur la cohomologie à supports compacts des variétés de Shimura pour  $GSp(4)/\mathbb{Q}$* , *Compositio Math.* **105** (1997), no. 3, 267–359.
- [Lau2] G. Laumon, *Fonctions zétas des variétés de Siegel de dimension trois*, in *Formes automorphes II: le cas du groupe  $GSp(4)$* , Edited by J. Tilouine, H. Carayol, M. Harris, M.-F. Vigneras, *Asterisque* **302**, *Soc. Math. France Astérisque* (2006).
- [MW] C. Moeglin and J.-L. Waldspurger, *Poles des fonctions  $L$  de paires pour  $GL(N)$* , Appendice, *Ann. Sci. École Norm. Sup.* (4) **22** (1989), 667–674.
- [Pic] *Zeta Functions of Picard Modular Surfaces*, edited by R.P. Langlands and D. Ramakrishnan, CRM Publications, Montréal (1992).
- [Ra1] D. Ramakrishnan, *Pure motives and automorphic forms*, in *Motives*, (1994) *Proc. Sympos. Pure Math.* 55, Part 2, AMS, Providence, RI, 411–446.

- [Ra2] D. Ramakrishnan, *Modularity of the Rankin-Selberg  $L$ -series, and Multiplicity one for  $SL(2)$* , Annals of Mathematics **152** (2000), 45–111.
- [Ra3] D. Ramakrishnan, *Modularity of solvable Artin representations of  $GO(4)$ -type*, IMRN **2002**, No. **1** (2002), 1–54.
- [Ra4] D. Ramakrishnan, *Algebraic cycles on Hilbert modular fourfolds and poles of  $L$ -functions*, in *Algebraic groups and arithmetic*, 221–274, Tata Inst. Fund. Res., Mumbai (2004).
- [Ra5] D. Ramakrishnan, *Irreducibility of  $\ell$ -adic associated to regular cusp forms on  $GL(4)/\mathbb{Q}$* , preprint (2004), being revised.
- [Ra-Sh] D. Ramakrishnan and F. Shahidi, *Siegel modular forms of genus 2 attached to elliptic curves*, preprint, submitted (2006)
- [Ra-W] D. Ramakrishnan and S. Wang, *A cuspidality criterion for the functorial product on  $GL(2) \times GL(3)$  with a cohomological application*, IMRN **2004**, No. **27**, 1355–1394.
- [Ri] K. Ribet, *Galois representations attached to eigenforms with Nebentypus*, in *Modular functions of one variable V*, pp. 17–51, Lecture Notes in Math. **601**, Springer, Berlin (1977).
- [Se] J.-P. Serre, *Abelian  $\ell$ -adic representations*, Research Notes in Mathematics **7**, A.K. Peters Ltd., Wellesley, MA (1998).
- [Sh1] F. Shahidi, *On the Ramanujan conjecture and the finiteness of poles for certain  $L$ -functions*, Ann. of Math. (2) **127** (1988), 547–584.
- [Sh2] F. Shahidi, *A proof of the Langlands conjecture on Plancherel measures; Complementary series for  $p$ -adic groups*, Ann. of Math. **132** (1990), 273–330.
- [So] D. Soudry, *On Langlands functoriality from classical groups to  $GL_n$* , in *Automorphic forms I*, Astérisque **298**, 335–390 (2005).
- [T] J. Tate, *Les conjectures de Stark sur les fonctions  $L$  d’Artin en  $s = 0$* , Lecture notes edited by D. Bernardi and N. Schappacher, Progress in Mathematics **47** (1984), Birkhäuser, Boston, MA.

- [Wei] R. Weissauer, *Four dimensional Galois representations*, in *Formes automorphes II: le cas du groupe  $GSp(4)$* , Edited by J. Tilouine, H. Carayol, M. Harris, M.-F. Vigneras, Asterisque **302**, Soc. Math. France *Astérisque* (2006).

Dinakar Ramakrishnan  
253-37 Caltech  
Pasadena, CA 91125, USA.  
dinakar@caltech.edu